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$\mathcal{N} = 1$ supersymmetric path-integral Poisson-Lie duality

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Abstract: We extend the path-integral formulation of Poisson-Lie duality given in [3] to $\mathcal{N} = 1$ supersymmetric σ-models. Using an explicit representation of the generators of the Drinfel'd double corresponding to $G \otimes \mathrm{U}(1)^{\mathrm{dimG}}$ we discuss an application to non-abelian duality. The paper also contains the relevant background and some comments on Poisson-Lie duality.

Keywords: Sigma Models, Supersymmetry and Duality, Superspaces.

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1. Introduction

Poisson-Lie (PL) duality [1] extends the notion of duality of σ-models to target spaces without isometries, replacing the isometry condition by the weaker PL condition. It includes ordinary abelian as well as non-abelian duality. After its construction two obvious extensions seemed desirable: a quantum formulation and an application to supersymmetric models.

The quantum setting has been treated in [2]–[4], of which the most relevant reference for our discussion is the path-integral treatment in [3]. Supersymmetric extensions have been discussed in [5]–[9].

In the present article we give an extension of the path-integral treatment of PL duality to $\mathcal{N}=1$ supersymmetric non-linear σ-models. We further comment on some questions of general relevance to PL duality, namely we give an explicit realization of the generators of the Drinfel'd double for non-abelian duality and we look for possible roles in PL duality for the extended currents that result when the Lie derivative of the background antisymmetric tensor field b satisfies $\mathcal{L}b = d\omega$ [10]. We also discuss the relation between $\mathcal{N}=1$ supersymmetric PL duality and non-abelian duality at the level of actions. To make the presentation more readable we have included background material on isometries, actions on group manifolds, non-abelian duality and PL duality. An introduction on duality which may serve as further background is found in [11, 12].

The outline of the paper is the following. We start by giving some background material on isometries and actions on group manifolds. Then in section 3 we give the basics of non-abelian duality. In section 4 we give a short summary of PL duality including the Drinfel'd double and the path-integral derivation of Tyurin and von Unge [3]. In section 5 we generalize this path-integral to $\mathcal{N}=1$ supersymmetry. The non-abelian limit of the $\mathcal{N}=1$ is presented in section 6 in where we explicitly construct a set of generators spanning the Drinfel'd double $G\otimes \mathrm{U}(1)^{\mathrm{dimG}}$, where G must be a compact group. We also find the gauge fixed action in the non-abelian limit. In two appendices we discuss a modified current as well as WZW models in the PL setting.

2. Actions and isometries

A string propagating in a curved background with metric g_{ij}, NS antisymmetric tensor field b_{ij} and dilaton ϕ, has a bosonic part given by the σ-model action

$$S=-\frac{1}{4\pi\alpha'}\int_W d^2\sigma\Big(\sqrt{-h}h^{mn}\partial_m x^i\partial_n x^j g_{ij}(x^k)+\epsilon^{mn}\partial_m x^i\partial_n x^j b_{ij}(x^k)-\\ -\alpha'\sqrt{-h}R^{(2)}\phi(x^k)\Big)\,, \tag{2.1}$$

where x^i ($i=1,\ldots,\mathrm{dim}T$) are the coordinates on the target space T (spacetime), h_{mn} is the auxiliary world-sheet metric, and $R^{(2)}$ the corresponding curvature. With application to strings in mind we thus study the σ-model (in units where $\alpha'=1/2\pi$)

$$S[x]=\int_W d^2\xi\partial x^i(g_{ij}+b_{ij})(x)\bar\partial x^j=\int_W d^2\xi\partial x^i f_{ij}(x)\bar\partial x^j\,, \tag{2.2}$$

which is (2.1) in conformal gauge, light-cone coordinates $\xi=\frac{1}{2}(\tau+\sigma)$ and $\bar\xi=\frac{1}{2}(\tau-\sigma)$ and with the dilaton field ϕ set to zero (later we comment briefly on non-zero ϕ).

We shall be interested in the case when (some of) the x^i's are acted on by a Lie-group G with Lie-algbra $\mathcal{G}$ generated by $\{t_a\}$

$$[t_a,t_b]=f_{ab}{}^c t_c\,. \tag{2.3}$$

The relevant part of (2.2) may then be written as

$$S[g]=\int_W d^2\xi(g^{-1}\partial g)^a E_{ab}(g)(g^{-1}\bar\partial g)^b\,, \tag{2.4}$$

where the correspondence is via the left and right invariant forms on $\mathcal{G}$

$$L=g^{-1}dg=t_a L^a{}_i dx^i\,;\qquad R=dgg^{-1}=t_a R^a{}_i dx^i\,. \tag{2.5}$$

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The coordinates x^i parametrize the group manifold $\mathcal{M}$, the group action is

$$\delta x^i = \epsilon^a R^i_a\,, \tag{2.6}$$

and

$$E_{ab} = L^i_a f_{ij} L^j_b\,. \tag{2.7}$$

For the special case of interest for strings when (2.1) corresponds to a WZW-model, the correspondence is

$$S[g] = \int_{\partial Y} d^2\xi\, \mathrm{Tr}(g^{-1}\partial g)(g^{-1}\bar\partial g) + \frac{1}{3}\int_Y \mathrm{Tr}(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg)\,, \tag{2.8}$$

with

$$g_{ij} = L^a_i \eta_{ab} L^b_j = R^a_i \eta_{ab} R^b_j\,, \tag{2.9}$$

and $\eta_{ab} = \mathrm{Tr}(t_a t_b)$ the Cartan-Killing metric. Further, the torsion is in this case given by the left or right invariant forms as $H_{ijk} = \frac{1}{2}L^a_i L^b_j L^c_k f_{abc} = \frac{1}{2}R^a_i R^b_j R^c_k f_{abc}$ which gives the "x"-version of the last term in (2.8) as

$$S_{WZW}[x] = \frac{1}{3!}\int_Y d^3y\epsilon^{\mu\nu\lambda} H_{ijk}\partial_\mu x^i \partial_\nu x^j \partial_\lambda x^k = \int_{W=\partial Y} d^2\xi \partial x^i b_{ij}\bar\partial x^j\,, \tag{2.10}$$

where we assume conditions such that the last integral is well-defined [13].

As a first example of a group G we consider (generalized) isometries of the σ-model (2.2). We thus take the right-invariant forms (say) to be Killing vectors $R^i_a = k^i_a$ and the algebra is $[k_a, k_b] = f_{ab}{}^c k_c$. The action (2.2) or (2.8) is now invariant under the group action (2.6), $\delta x^i = \epsilon^a k^i_a$, due to the generalized isometry conditions

$$\mathcal{L}_{k_a} g_{ij} = 0\,, \qquad \mathcal{L}_{k_a} H_{ijk} = 0\,. \tag{2.11}$$

The latter of these is equivalent to [10]

$$\mathcal{L}_{k_a} b_{ij} = \partial_{[i}\omega_{j]a}\,, \tag{2.12}$$

for some one-forms ω_{ia}. Note that b_{ij} is defined only up to (spacetime) gauge transformations $b_{ij} \to b_{ij} + \partial_{[i}\lambda_{j]}$, where λ_j are the components of some one-form λ. Under this gauge transformation the ω_{ai} transform as $\omega_{ai} \to \omega_{ai} + \mathcal{L}_{k_a}\lambda_i$.

For the WZW-model the WZW-term (2.10) varies into

$$\delta S_{WZW} = \frac{1}{2!}\int_W d^2\sigma \epsilon^a k^i_a H_{ijk}\partial_\mu x^j \partial_\nu x^k \epsilon^{\mu\nu}\,. \tag{2.13}$$

This integral vanishes (up to a surface term) if $k^i_a H_{ijk} = \partial_{[j} v_{k]a}$ [13], where v_{ia} is a component of a 1-form. This is again satisfied under (2.11) and (2.12).

In general the Noether currents corresponding to the symmetries generated by k^i_a are [10]

$$\hat{J}_a = \partial x^i f_{ij} k^j_a + \omega_{ai}\partial x^i\,; \qquad \hat{\bar{J}}_a = k^i_a f_{ij}\bar\partial x^j - \omega_{ai}\bar\partial x^i\,. \tag{2.14}$$

The equation of motion state that this current is conserved

$$\partial \hat{\bar{J}}_a + \bar\partial \hat{J}_a = 0\,. \tag{2.15}$$

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3. Non-abelian dualization

When a σ-model has generalized isometries as described above it may be dualized [12, 14]. In this section we give a brief description of this non-abelian dualization. First the fields x^i in (2.2) are separated into those acted on by the isometry $x^{\hat{i}}$ and those not acted on x^α (spectators)

$$S[x^{\hat{i}}, x^\alpha] = \int d^2\xi \left[\partial x^{\hat{i}} f_{\hat{i}\hat{j}} \bar{\partial} x^{\hat{j}} + \partial x^{\hat{i}} f_{\hat{i}\beta} \bar{\partial} x^\beta + \partial x^\alpha f_{\alpha\hat{j}} \bar{\partial} x^{\hat{j}} + \partial x^\alpha f_{\alpha\beta} \bar{\partial} x^\beta\right]. \tag{3.1}$$

Secondly, the isometry

$$\delta x^{\hat{i}} = \epsilon^a(\xi, \bar{\xi}) k_a^{\hat{i}}(x^{\hat{j}}, x^\alpha) \tag{3.2}$$

is gauged. The gauging of the symmetric part of the action (3.1) by minimal coupling is straight forward, but to gauge the antisymmetric part by minimal coupling the torsion potential must be well-defined and Lie-invariant (i.e. $\mathcal{L}_{k_a} b_{ij} = 0$) [13]. Letting $\partial x^{\hat{i}} \to \partial x^{\hat{i}} + A^a L_a^{\hat{i}}$ and $\bar{\partial} x^{\hat{i}} \to \bar{\partial} x^{\hat{i}} + \bar{A}^a L_a^{\hat{i}}$ gives the gauged version of (3.1). A first order action is found by fixing a gauge ($\partial x^{\hat{i}} = \bar{\partial} x^{\hat{i}} = 0$), and adding a term including Lagrange multipliers

$$\begin{aligned} S^{(1)}[x, A, \lambda] = \int d^2\xi \left[A^a E_{ab} \bar{A}^b + A^a F^R_{a\beta} \bar{\partial} x^\beta + \partial x^\alpha F^L_{\alpha b} \bar{A}^b + \partial x^\alpha f_{\alpha\beta} \bar{\partial} x^\beta\right] + \\ + \int d^2\xi \lambda_a f^a \,, \end{aligned} \tag{3.3}$$

where we have defined $E_{ab} = L_a^{\hat{i}} f_{\hat{i}\hat{j}} L_b^{\hat{j}}$, $F^R_{a\beta} = L_a^{\hat{i}} f_{\hat{i}\beta}$ and $F^L_{\alpha b} = f_{\alpha\hat{j}} L_b^{\hat{j}}$. Here λ_a are Lagrange multipliers that take values in the Lie-algebra $\mathcal{G}$ and transforms in the adjoint representation. The field strength is $f^a = \partial \bar{A}^a - \bar{\partial} A^a + f^a_{bc} A^b \bar{A}^c$ where the gauge-fields A^a, $\bar{A}^a$ take values in the Lie algebra of the isometry group.

Varying the first-order action w.r.t. λ_a one obtains, at least locally, $A^a = (g^{-1}\partial g)^a$, $\bar{A}^a = (g^{-1}\bar{\partial} g)^a$. So modulo global issues [12] the first order action goes into the original one when substituting this solution. If one instead integrates out the gauge fields A^a, $\bar{A}^a$ one should get the "dual" action. However, the symmetry of the background that is used in the abelian case to dualize back to the original action is non-local in the non-abelian case. Dualization of this non-local symmetry does not recover the original action [12].

In the abelian case where k_a^i commute the original f_{ij} and dual $\tilde{f}_{ij}$ backgrounds are related via the Buscher transformation [15]

$$\begin{array}{ll} \tilde{f}_{\hat{i}\hat{j}} = (f^{-1})_{\hat{i}\hat{j}}\,; & \tilde{f}_{\hat{i}\beta} = (f^{-1})_{\hat{i}}{}^{\hat{j}} f_{\hat{j}\beta} \\ \tilde{f}_{\alpha\hat{j}} = -f_{\alpha\hat{i}} (f^{-1})^{\hat{i}}{}_{\hat{j}} & \tilde{f}_{\alpha\beta} = f_{\alpha\beta} - f_{\alpha\hat{i}} (f^{-1})^{\hat{i}\hat{j}} f_{\hat{j}\beta}\,. \end{array} \tag{3.4}$$

The transformation maps manifolds without torsion ($f_{\hat{i}\beta} = f_{\beta\hat{i}}$) on to manifolds with torsion ($\tilde{f}_{\hat{i}\beta} = -\tilde{f}_{\beta\hat{i}}$).

4. Poisson-Lie duality

A more general and useful scheme for finding dual actions, not based on the existence of generalized isometries, is the PL duality [1] where the isometry is replaced with a weaker condition. This duality is most easily discussed in terms of the would-be Noether currents of the transformations $\delta x^i = \epsilon^a R^i_a$;

$$J_a = \partial x^i f_{ij} R^j_a \, ; \qquad \bar{J}_a = R^i_a f_{ij} \bar{\partial} x^j \, . \tag{4.1}$$

Since $\mathcal{L}_{R_a} f_{ij} \neq 0$ these transformations are no-longer a symmetry of the action (for constant ϵ^a), in fact

$$\delta S = \int d^2\xi \partial x^i \bar{\partial} x^j \epsilon^a (\mathcal{L}_{R_a} f_{ij}) \, , \tag{4.2}$$

and the field equations

$$\partial \bar{J}_a + \bar{\partial} J_a - \mathcal{L}_{R_a} f_{ij} \partial x^i \bar{\partial} x^j = 0 \, , \tag{4.3}$$

no-longer look like Bianchi identities. However, Klimčik and Ševera [1] introduced the following generalization of the isometry condition for the background

$$\mathcal{L}_{R_a} f_{\hat{i}\hat{j}} = - f_{\hat{i}\hat{k}} R^{\hat{k}}_b \tilde{f}^{bc}_a R^{\hat{l}}_c f_{\hat{l}\hat{j}} \, , \tag{4.4}$$

which turns (4.3) into

$$\partial \bar{J}_a + \bar{\partial} J_a + J_b \tilde{f}^{bc}_a \bar{J}_c = 0 \, . \tag{4.5}$$

Here $\tilde{f}^{ab}_c$ are structure constants in a dual Lie algebra, and Klimčik and Ševera went on to show that the condition (4.3) can be solved and the dual model found provided that the Lie algebra $\mathcal{G}$ and its dual $\tilde{\mathcal{G}}$ form what is called a Drinfel'd double [16]–[18] which we now briefly decribe.

Let G and $\tilde{G}$ be symmetry groups of the original σ-model and the dual one, respectively, with $\dim G = \dim \tilde{G}$. The corresponding Lie algebras are $\mathcal{G}$ and $\tilde{\mathcal{G}}$. Then the Drinfel'd double $D \equiv G \otimes \tilde{G}$ and comes equipped with an invariant inner product $\langle \; , \; \rangle$ and the corresponding algebra $\mathcal{D}$ consists of the two subalgebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ that are null-spaces w.r.t. this product. We choose two sets of generators $\{T_a\}$ and $\{T^a\}$ so that $\{T_a\}$ span $\mathcal{G}$ and $\{T^a\}$ span $\tilde{\mathcal{G}}$. The set $T_A \in \{T_a, T^b\}$ then span $\mathcal{D}$. The Lie algebra of the Drinfel'd double generated by T_a and T^a ($a = 1, \ldots, \mathrm{dimG}$), is

$$\begin{aligned} [T_a, T_b] &= f^c_{ab} T_c \, , \\ [T^a, T^b] &= \tilde{f}^{ab}_c T^c \, , \\ [T_a, T^b] &= \tilde{f}^{bc}_a T_c - f^b_{ac} T^c \, , \end{aligned} \tag{4.6}$$

where f^c_{ab} and $\tilde{f}^{ab}_c$ are the structure constants of $\mathcal{G}$ and $\tilde{\mathcal{G}}$, respectively, and satisfy the bi-Lie algebra $(\mathcal{G}, \tilde{\mathcal{G}})$ consistency condition

$$f^a_{dc} \tilde{f}^{rs}_a = \tilde{f}^{as}_c f^r_{da} + \tilde{f}^{ra}_c f^s_{da} - \tilde{f}^{as}_d f^r_{ca} - \tilde{f}^{ra}_d f^s_{ca} \, . \tag{4.7}$$

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This condition arises in the PL duality context as the condition $[\mathcal{L}_{k_a}, \mathcal{L}_{k_b}] = f_{ab}{}^c \mathcal{L}_{k_c}$ applied to (4.4). The invariant inner product between the generators has the following properties

$$\langle T_a, T_b \rangle = \langle T^a, T^b \rangle = 0\,, \qquad \langle T_a, T^b \rangle = \delta_a{}^b \tag{4.8}$$

and obeys the invariance condition

$$\langle \mathcal{X} T^A \mathcal{X}^{-1}, T^B \rangle = \langle T^A, \mathcal{X}^{-1} T^B \mathcal{X} \rangle\,, \tag{4.9}$$

where $\mathcal{X}$ is any element of the Drinfel'd double or one of its subgroups.

We define,

$$\begin{aligned} \mu^{ab}(g) &= \langle g T^a g^{-1}, T^b \rangle\,; & \nu^a{}_b(g) &= \langle g T^a g^{-1}, T_b \rangle \\ \alpha_b{}^a(\tilde{g}) &= \langle \tilde{g} T_b \tilde{g}^{-1}, T^a \rangle\,; & \beta_{ab}(\tilde{g}) &= \langle \tilde{g} T_a \tilde{g}^{-1}, T_b \rangle \end{aligned} \tag{4.10}$$

which obey $\mu(g^{-1}) = \mu^t(g)$, $\nu(g^{-1}) = \nu^{-1}(g)$, $\alpha(\tilde{g}^{-1}) = \alpha^{-1}(\tilde{g})$ and $\beta(\tilde{g}^{-1}) = \beta^t(\tilde{g})$ where t stands for transpose.

We now return to the solution of (4.4) given by Klimčik and Ševera. With $f_{\hat{i}\hat{j}} = L^a_{\hat{i}} E_{ab} L^b_{\hat{j}}$ as in (2.7) the solution is

$$E_{ab} = ((E^0)^{-1} + \Pi)^{-1}_{ab}\,; \qquad \Pi^{ab} = \mu^{ac} \nu_c{}^b\,. \tag{4.11}$$

Similarly, in the dual theory one has relations corresponding to (4.4) and (4.5) and

$$\tilde{E}^{ab} = [(E^0 + \tilde{\Pi})^{-1}]^{ab}\,; \qquad \tilde{\Pi}_{ab} = \beta_{ac} \alpha^c{}_b\,. \tag{4.12}$$

The abelian and non-abelian dualities described previously are special cases of the more general PL duality. In the non-abelian case we have $\mu^{ab} = 0$, $\alpha^a{}_b = \delta^a{}_b$ and $\beta_{ab} = f^c_{ab} \tilde{x}_c$, where $\tilde{x}_c$ is the dual non-inert coordinates, so that $E_{ab} = E^0_{ab}$ and $\tilde{E}^{ab} = [(E^0 + f^c \tilde{x}_c)^{-1}]^{ab}$.

As described above PL duality acts at the classical level. An important step towards the quantum implementation was taken in [3], where it was shown how PL duality can be derived from a constrained WZW-model defined on the Drinfel'd double D, starting from the path-integral

$$\mathcal{Z} = \int \mathcal{D}l \mathcal{D}x\, \delta[\langle l^{-1}\partial l, T^a \rangle E^0_{ab} + \partial x^\alpha F^L_{\alpha b} - \langle l^{-1}\partial l, T_b \rangle] e^{-I[l,x]}\,, \tag{4.13}$$

where

$$I[l,x] = I[l] + \int d^2\xi [\langle l^{-1}\partial l, T^a \rangle F^R_{a\alpha} \bar{\partial} x^\alpha + \partial x^\alpha \hat{f}_{\alpha\beta} \bar{\partial} x^\beta]\,, \tag{4.14}$$

and $l \in D$. It should be stressed that $\hat{f}_{\alpha\beta}$ is generally not equal to $f_{\alpha\beta}$ that appears in (3.3).[1] The WZW-model $I[l]$ on the Drinfel'd double can be written

$$I[l] = \int d^2\xi \langle l^{-1}\partial l, l^{-1}\bar{\partial} l \rangle + \int d^3 y \langle l^{-1}\partial_t l, [l^{-1}\partial l, l^{-1}\bar{\partial} l] \rangle\,. \tag{4.15}$$

[1] In [3] this is not clear.

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We also note that the path-integral (4.13) can be obtained from the path-integral

$$\mathcal{Z} = \int \mathcal{D}l\mathcal{D}x\mathcal{D}\bar{c}e^{-I[l,x,\bar{c}]}\,, \tag{4.16}$$

where

$$I[l,x,\bar{c}] = I[l,x] + \int d^2\xi[\langle l^{-1}\partial l, T^a\rangle E^0_{ab} + \partial x^\alpha F^L_{\alpha b} - \langle l^{-1}\partial l, T_b\rangle]\bar{c}^b \tag{4.17}$$

by integrating out the Lagrange multiplier $\bar{c}^b$ (We use (4.17) in section 6 where we study the $\mathcal{N}=1$ supersymmetric generalization of non-abelian duality).

The dualization process goes as follows. The original action is recovered when the group element l in (4.13) is decomposed as $l = \tilde{h}g$. This makes the left-invariant Haar measure split into $\mathcal{D}(\tilde{h}g) = \mathcal{D}\tilde{h}\mathcal{D}g\det(\nu^{-1})$, where $\nu \equiv \nu^a{}_b$ is defined in eq. (4.10). By letting $\mathrm{Tr} \to \langle\ ,\ \rangle$ in the Polyakov-Wiegmann formula [19]

$$S[g_1g_2] = S[g_1] + S[g_2] + \int d^2\xi\, \mathrm{Tr}(g_1^{-1}\partial g_1\bar{\partial}g_2g_2^{-1}) \tag{4.18}$$

and using the inner product defined in (4.8) the two first terms of $I[\tilde{h}g]$, namely $I[\tilde{h}]$ and $I[g]$, are zero. Integrating out $\tilde{h}$ from the remaining part of $I[\tilde{h}g]$ gives back the original action (3.1). When we decompose l as $l = h\tilde{g}$ and integrate out h we obtain the dual action.

This procedure gives non-trivial jacobians that must be regularized in the quantum theory. Using heat kernel regularization, it was shown in [3] that as a result the dilaton in both the original and the dual theory gets an extra shift. In the original theory the shift is

$$\phi = \phi^0 + \ln\det E(g,x^\alpha) - \ln\det E^0(x^\alpha) \tag{4.19}$$

while in the dual theory it transforms as

$$\tilde{\phi} = \phi^0 + \ln\det \tilde{E}(\tilde{g},x^\alpha)\,. \tag{4.20}$$

In the non-abelian limit when $E(g,x^\alpha) = E^0(x^\alpha)$ we see that we have the usual transformation law between ϕ and $\tilde{\phi}$, namely

$$\tilde{\phi} = \phi + \ln\det \tilde{E}(\tilde{g},x^\alpha)\,, \tag{4.21}$$

where $\tilde{E}^{ab} = [(E^0 + f^c\tilde{x}_c)^{-1}]^{ab}$.

5. $\mathcal{N}=1$ supersymmetric Poisson-Lie duality

In this section we present PL duality in a supersymmetric setting. For $(1,0)$ and $(1,1)$ supersymmetry this is discussed at the classical level in [7]. The path-integral formulation is new.

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The presence of one supersymmetry on the world-sheet introduces additional spinors that transform under the action of the Drinfel'd double but otherwise changes very little. This is similar to the non-abelian duality, where the gauging of isometries needed follow essentially the bosonic pattern for $\mathcal{N}=1$ [20]. Hence, the presentation of the basics of the Drinfel'd double given above can be taken over to the $\mathcal{N}=1$ case. However the elements of the Drinfel'd double D and its subgroups G and $\tilde{G}$ are now real $\mathcal{N}=1$ superfields.

The $\mathcal{N}=1$ generalization of the bosonic action in (2.2) is[2] [21]

$$S=-\frac{1}{4}\int d^2\sigma d^2\theta\left[G_{ij}(X)D^AX^iD_AX^j-B_{ij}(X)D^AX^i(\gamma D)_AX^j\right], \tag{5.1}$$

where A,B are spinor indices, G_{ij} and B_{ij} are the metric and the torsion potential, respectively, and X^i ($i=1,\ldots,\dim T$) are real scalar superfields. The θ-independent components of X^i is x^i. The θ-independent components of G_{ij} and B_{ij} are g_{ij} and b_{ij}, respectively. For our discussion it is convenient to rewrite the action using the $\pm$-notation. Seperating out the spectators (X^α) we get

$$\begin{aligned}S=i\int d^2\xi d^2\theta\Big[&D_+X^{\hat{i}}F_{\hat{i}\hat{j}}D_-X^{\hat{j}}+D_+X^{\hat{i}}F_{\hat{i}\beta}D_-X^\beta+D_+X^\alpha F_{\alpha\hat{j}}D_-X^{\hat{j}}+\\&+D_+X^\alpha F_{\alpha\beta}D_-X^\beta\Big],\end{aligned} \tag{5.2}$$

where $F_{ij}=G_{ij}+B_{ij}$ and $i=(\hat{i},\alpha)$.

Again we want to study the case when the $x^{\hat{i}}$ transform under some group G, and thus study the σ-model on some group manifold $\mathcal{M}$. The fields are $\mathcal{N}=1$ real scalar group-valued superfield, $U\in G$, whose θ-independent part is g, and the spectators X^α which are $\mathcal{N}=1$ real superfields. The corresponding action on $\mathcal{M}$ may be written

$$\begin{aligned}S=i\int d^2\xi d^2\theta\Big[&(U^{-1}D_+U)^aE_{ab}(U^{-1}D_-U)^b+(U^{-1}D_+U)^aF^R_{a\beta}D_-X^\beta\\&+D_+X^\alpha F^L_{\alpha b}(U^{-1}D_-U)^b+D_+X^\alpha F_{\alpha\beta}D_-X^\beta\Big].\end{aligned} \tag{5.3}$$

To make contact between (5.2) and (5.3) we follow the same route as in the bosonic case. The superfields $X^{\hat{i}}$ in (5.2) are thus taken to transform under G and are related to U via left invariant one-form $U^{-1}D_\pm U=T_aL^a{}_{\hat{i}}D_\pm X^{\hat{i}}$. Varying the action under

$$\delta X^{\hat{i}}=\epsilon^a(\xi,\bar{\xi},\theta)R^{\hat{i}}_a(X^{\hat{j}},X^\alpha) \tag{5.4}$$

gives the field equations

$$D_{(+}J_{a-)}-\sum_{i=(\hat{i},\alpha)}\mathcal{L}_{R_a}F_{ij}D_+X^iD_-X^j=0\,, \tag{5.5}$$

[2] We write the algebra as $\{D_A,D_B\}=2i\partial_{AB}$; $\partial_{AB}=(\gamma^m)_{AB}\partial_m$, where $(\gamma^m)_A{}^B=(\sigma^2,-i\sigma^1)$. We further define $(\gamma)_A{}^B=\sigma^3$ and we raise and lower indices using $C_{AB}=-C^{AB}=\sigma^2$.

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where the "currents"

$$J_{a+} = -(D_+X^{\hat{i}}F_{\hat{i}\hat{j}} + D_+X^{\alpha}F_{\alpha\hat{j}})R_a^{\hat{j}}\,; \qquad J_{a-} = R_a^{\hat{i}}(F_{\hat{i}\hat{j}}D_-X^{\hat{j}} + F_{\hat{i}\beta}D_-X^{\beta})\,, \tag{5.6}$$

are Noether currents when $R_a^{\hat{i}}$ generates an isometry. For these currents to satisfy a Maurer-Cartan equation we again introduce a PL condition. The supersymmetric version of the Maurer-Cartan equation (4.5) is

$$D_{(+}J_{a-)} - J_{b+}\tilde{f}_a^{bc}J_{c-} = 0\,, \tag{5.7}$$

and the PL condition for the background that ensures that (5.5) is equivalent to (5.7) are

$$\begin{aligned} \mathcal{L}_{R_a}F_{\hat{i}\hat{j}} &= -F_{\hat{i}\hat{k}}R_b^{\hat{k}}\tilde{f}_a^{bc}R_c^{\hat{l}}F_{\hat{l}\hat{j}}\,; & \mathcal{L}_{R_a}F_{\hat{i}\beta} &= -F_{\hat{i}\hat{k}}R_b^{\hat{k}}\tilde{f}_a^{bc}R_c^{\hat{l}}F_{\hat{l}\beta} \\ \mathcal{L}_{R_a}F_{\alpha\hat{j}} &= -F_{\alpha\hat{k}}R_b^{\hat{k}}\tilde{f}_a^{bc}R_c^{\hat{l}}F_{\hat{l}\hat{j}}\,; & \mathcal{L}_{R_a}F_{\alpha\beta} &= -F_{\alpha\hat{j}}R_b^{\hat{j}}\tilde{f}_a^{bc}R_c^{\hat{l}}F_{\hat{l}\beta}\,, \end{aligned} \tag{5.8}$$

which is the supersymmetric version of (4.4).

Now we want to extend the analysis given in [3] to $\mathcal{N}=1$ supersymmetry. A direct generalization of (4.13) to $\mathcal{N}=1$ supersymmetry is the following constrained $\mathcal{N}=1$ functional

$$\mathcal{Z} = \int \mathcal{D}L\mathcal{D}X\,\delta[\langle L^{-1}D_+L, T^a\rangle E^0_{ab} + D_+X^{\alpha}F^L_{\alpha b} - \langle L^{-1}D_+L, T_b\rangle]\exp(-I[L,X])\,, \tag{5.9}$$

where the action

$$I[L,X] = I[L] + i\int d^2\xi d^2\theta\left[\langle L^{-1}D_+L, T^a\rangle F^R_{a\beta}D_-X^{\beta} + D_+X^{\alpha}\hat{F}_{\alpha\beta}D_-X^{\beta}\right], \tag{5.10}$$

and where $I[L]$ is an $\mathcal{N}=1$ WZW model on the Drinfel'd double. Here E^0_{ab}, $F^L_{\alpha b}$, $F^R_{a\beta}$ and $\hat{F}_{\alpha\beta}$ depend on the spectator superfields X^{α}.

The element $L \in D$ in (5.9) can be decomposed near the identity in two ways; $L = \tilde{V}U = V\tilde{U}$, where $U \in G$ and $\tilde{V} \in \tilde{G}$. To derive the original action we decompose L as $\tilde{V}U$. After integrating out $\tilde{V}$ via a change of variables we can read off the metric and the torsion potential:

$$\begin{aligned} F_{\hat{i}\hat{j}} &= L^a_{\hat{i}}E_{ab}L^b_{\hat{j}}\,; \\ F_{\hat{i}\beta} &= L^a_{\hat{i}}E_{ab}(E^0_{bc})^{-1}F^R_{c\beta}\,; \\ F_{\alpha\hat{j}} &= F^L_{\alpha a}(E^0_{ab})^{-1}E_{bc}L^c_{\hat{j}}\,; \\ F_{\alpha\beta} &= \hat{F}_{\alpha\beta} + F^L_{\alpha a}((E^0)^{-1}E(E^0)^{-1} - (E^0)^{-1})^{ab}F^R_{b\beta}\,. \end{aligned} \tag{5.11}$$

The dual theory is found by decomposing L as $V\tilde{U}$. When we integrate out V again by changing variables we find the background

$$\begin{aligned} \tilde{F}^{\hat{i}\hat{j}} &= \tilde{L}^{\hat{i}}_a\tilde{E}^{ab}\tilde{L}^{\hat{j}}_b\,; \\ \tilde{F}^{\hat{i}}{}_{\beta} &= \tilde{L}^{\hat{i}}_a\tilde{E}^{ab}F^R_{b\beta}\,; \\ \tilde{F}_{\alpha}{}^{\hat{j}} &= -F^L_{\alpha a}\tilde{E}^{ab}\tilde{L}^{\hat{j}}_b\,; \\ \tilde{F}_{\alpha\beta} &= \hat{F}_{\alpha\beta} - F^L_{\alpha a}\tilde{E}^{ab}F^R_{b\beta}\,. \end{aligned} \tag{5.12}$$

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The generalized Buscher transformation are

$$
\begin{aligned}
\tilde{E}^{-1} - \beta\alpha = (E^{-1} + \mu\nu)^{-1} &= E^{0}(x^{\alpha})\,; \\
\tilde{E}^{-1}\tilde{\mathcal{F}}^{R} = E^{0}E^{-1}\mathcal{F}^{R} &= F^{R}\,; \\
-\tilde{\mathcal{F}}^{L}\tilde{E}^{-1} = \mathcal{F}^{L}E^{-1}E^{0} &= F^{L}\,; \\
\tilde{F} - \tilde{\mathcal{F}}^{L}\tilde{E}^{-1}\tilde{\mathcal{F}}^{R} = F + \mathcal{F}^{L}(E^{-1}E^{0}E^{-1} - E^{-1})\mathcal{F}^{R} &= \hat{F}\,,
\end{aligned}
\tag{5.13}
$$

where $\mathcal{F}^{L}_{\alpha b} \equiv F_{\alpha\hat{j}}L^{\hat{j}}{}_{b}$, $\mathcal{F}^{R}_{a\beta} \equiv L_{a}{}^{\hat{i}}F_{\hat{i}\beta}$, $\tilde{\mathcal{F}}^{Lb}_{\alpha} \equiv \tilde{F}_{\alpha}{}^{\hat{j}}\tilde{L}_{\hat{j}}{}^{b}$ and $\tilde{\mathcal{F}}^{Ra}{}_{\beta} \equiv \tilde{L}^{a}{}_{\hat{i}}\tilde{F}^{\hat{i}}{}_{\beta}$. The bosonic version of these generalized Buscher transformation was given in this form in [3] and, earlier in a different form in [1]. In the abelian case where $\beta = \mu = 0$ a combination of eq.(5.11) and (5.12) gives eq. (3.4).

We end this section by showing how the $\mathcal{N} = 1$ dilaton superfield Φ transforms under PL duality. Since the transformation of Φ is analogous to the transformation of the bosonic component ϕ given in eqs.(4.19)-(4.20) we just give the transformation law for the fermionic component

$$
\begin{aligned}
\rho_{\pm} &= \rho^{0}_{\pm} + M_{U}|\lambda_{\pm} - (M_{\alpha}| - N_{\alpha}|)\eta^{\alpha}_{\pm} \\
\tilde{\rho}_{\pm} &= \rho^{0}_{\pm} + \tilde{M}_{\tilde{U}}|\tilde{\lambda}_{\pm} + \tilde{M}_{\alpha}|\eta^{\alpha}_{\pm}
\end{aligned}
\tag{5.14}
$$

and the auxiliary field

$$
\begin{aligned}
W &= W^{0} + iM_{UU}|\lambda_{-}\lambda_{+} + M_{U}|Y + iM_{\alpha U}|\eta^{\alpha}_{[-}\lambda_{+]} + i(M_{\alpha\beta}| - N_{\alpha\beta}|)\eta^{\alpha}_{-}\eta^{\beta}_{+} - N_{\alpha}T^{\alpha} \\
\tilde{W} &= W^{0} + i\tilde{M}_{\tilde{U}\tilde{U}}|\tilde{\lambda}_{-}\tilde{\lambda}_{+} + i\tilde{M}_{\alpha\tilde{U}}|\eta^{\alpha}_{[-}\tilde{\lambda}_{+]} + i\tilde{M}_{\alpha\beta}|\eta^{\alpha}_{-}\eta^{\beta}_{+} + \tilde{M}_{\tilde{U}}|\tilde{Y} + \tilde{M}_{\alpha}|T^{\alpha}\,.
\end{aligned}
\tag{5.15}
$$

Here we have defined $M(U, X^{\alpha}) = \ln\det E(U, X^{\alpha})$, $N(X^{\alpha}) = \ln\det E^{0}(X^{\alpha})$ and M_{U} means the derivative of M w.r.t. U and N_{α} means the derivative of N w.r.t. the spectator field X^{α} and $|$ means "the θ-independent component of". The component fields are defined as follows: $\lambda_{\pm} = D_{\pm}U|$, $Y = D^{2}U|$, $\rho_{\pm} = D_{\pm}\Phi|$, $W = D^{2}\Phi|$, $\eta^{\alpha}_{\pm} = D_{\pm}X^{\alpha}|$ and $T^{\alpha} = D^{2}X^{\alpha}|$. The dual components are defined similarly.

6. $\mathcal{N} = 1$ non-abelian duality

In this section we give an explicit representation of the generators of the Drinfel'd double relevant for non-abelian duality. We also discuss non-abelian duality for $\mathcal{N} = 1$ supersymmetric models in the PL path-integral setting, extending the bosonic analysis [3].

In the PL setting non-abelian duality corresponds to the dual group $\tilde{G}$ being abelian [1]. Correspondingly we choose a Drinfel'd double $G \otimes \mathrm{U}(1)^{\mathrm{dimG}}$, where $\{T_{a}\}$ span $\mathcal{G}$ and $\{T^{a}\}$ span $\tilde{\mathcal{G}} = \sum_{1}^{\mathrm{dimG}} \oplus u(1)$; $(a = 1, \ldots, \mathrm{dimG})$ and the two sets of generators satisfy the algebra (ref. eq. 4.6)

$$
[T_{a}, T_{b}] = f^{c}_{ab}T_{c}\,; \qquad [T^{a}, T^{b}] = 0\,; \qquad [T_{a}, T^{b}] = -f^{b}_{ac}T^{c}\,. \tag{6.1}
$$

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An explicit representation of the set $\{T_a, T^a\}$ is

$$T_a = \begin{pmatrix} t_a & 0 \\ 0 & t_a \end{pmatrix}, \qquad T^a = \begin{pmatrix} 0 & \frac{1}{\lambda} t_a \\ 0 & 0 \end{pmatrix}, \tag{6.2}$$

where $t_a \in \mathcal{G}$ satisfy the algebra $[t_a, t_b] = f_{ab}{}^c t_c$ and $\mathrm{Tr}(t_a t_b) = \lambda \delta_{ab}$. Consistency requires that the structure constants are completely antisymmetric. Thus this representation is possible only when the group G is compact.

Let X and Y be two 2×2 block matrices where each block is a $r \times r$ matrix. We define the inner product (4.8) as

$$\langle X, Y \rangle = \mathrm{Tr}[XY]_{12}\,, \tag{6.3}$$

where $[\;\;]_{12}$ stands for the one-two block of the product XY. Since $\tilde{f}^{ab}{}_c = 0$, the bi-Lie algebra consistency condition (4.7) is trivial fulfilled.

Writing the group element of G as $\exp(X^a T_a)$ (and a similar parametrization of the dual group elements), we find

$$U = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \qquad \tilde{U} = \begin{pmatrix} \mathbf{1} & \frac{1}{\lambda}\tilde{X}_a t_a \\ 0 & \mathbf{1} \end{pmatrix}, \tag{6.4}$$

where $\mathbf{1}$ is the $r \times r$ identity matrix and $u = \exp(X^a t_a) \in G$. Moreover we find

$$\begin{aligned} &\mu^{ab}(U) = \mu^{ab}(U^{-1}) = 0\,; && \nu^a{}_b(U) = \nu^b{}_a(U^{-1}) = \tfrac{1}{\lambda}\,\mathrm{Tr}(u t_a u^{-1} t_b) \\ &\alpha_a{}^b(\tilde{U}) = \alpha_a{}^b(\tilde{U}^{-1}) = \delta_{ab}\,; && \beta_{ab}(\tilde{U}) = -\beta_{ab}(\tilde{U}^{-1}) = \tilde{X}_c f^b_{ca} = f_{ab}{}^c \tilde{X}_c \end{aligned} \tag{6.5}$$

From these relations it is easy to find the backgrounds

$$E_{ab}(u) = E^0_{ab}\,; \qquad \tilde{E}^{ab}(\tilde{u}) = [(E^0 + f^c \tilde{X}_c)^{-1}]^{ab}\,, \tag{6.6}$$

where $f^c \tilde{X}_c \equiv f_{ab}{}^c \tilde{X}_c$. The $\mathcal{N}=1$ non-abelian Buscher transformation can be found from eq. (5.13) by setting $E = E^0$, $\mu = 0$ and $\alpha = \mathbf{1}$.

We now turn to a comparison to the usual non-abelian duality formulation. A $\mathcal{N}=1$ supersymmetric generalizing of (4.17), (for simplicity we will set the spectators to zero in this section) is

$$I[L, C_-] = I[L] + i\int d^2\xi d^2\theta \left[\langle L^{-1}D_+L, T^a\rangle E^0_{ab} - \langle L^{-1}D_+L, T_b\rangle\right] C^b_-\,, \tag{6.7}$$

where C^b_- is a Lagrange multiplier. From (6.7) we may recover the $\mathcal{N}=1$ formulation of the traditional non-abelian duality in the form it has after gauge-fixing the coordinates on G to zero. Decomposing $L = V\tilde{U} \in D$, $V \in G$, $\tilde{U} \in \tilde{G}$, the action becomes

$$I[A_+, \tilde{J}_\pm, \tilde{X}, C_-] = i\int d^2\xi d^2\theta [A^a_+ (\alpha^{-1})_a{}^b \tilde{J}_{-b} + (A^a_+(\alpha^{-1})_a{}^c E^0_{cb} - A^a_+ \beta^t_{ab} - \tilde{J}_{+b}) C^b_-]\,, \tag{6.8}$$

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where $A_+ \equiv V^{-1}D_+V$ and $\tilde{J}_\pm \equiv \tilde{U}^{-1}D_\pm\tilde{U}$. After using eq. (6.5), and using the abelian current $\tilde{J}_{\pm a} = D_\pm\tilde{X}_a$, the action (6.8) may be written

$$I[A_+, C_-, \tilde{X}_a] = i\int d^2\xi d^2\theta[(A_+^a)D_-\tilde{X}_a - (D_+\tilde{X}_a)C_-^a + A_+^a(E_{ab}^0 + f_{ab}^c\tilde{X}_c)C_-^b] \,. \tag{6.9}$$

This is the first order action (up to a total derivative) one usually starts with to obtain the dual model in the $\mathcal{N}=1$ non-abelian duality. The action (6.9) needs some comments. First of all the components of the gauge field are A_+^a and C_-^a, where C_-^a as we remember was from the beginning a Lagrange multiplier. The "new" Lagrange multiplier is $\tilde{X}_a$ [3]. Note that in the abelian limit the terms involving the Lagrange multiplier

$$i\int d^2\xi d^2\theta[(A_+^a)D_-\tilde{X}_a - (D_+\tilde{X}_a)C_-^a] = i\int d^2\xi d^2\theta\tilde{X}_aF^a + \text{surface term}\,, \tag{6.10}$$

where $F^a = D_-A_+^a + D_+C_-^a$. This is exactly the correct term one needs to be sure to get back the original action when the Lagrange multiplier is integrated out [11, 12].

After a partial integration and variation of I w.r.t. $\tilde{X}$, we get the zero "field strength" condition

$$D_-A_+^a + D_+C_-^a + A_+^bf_{bc}^aC_-^c = 0\,. \tag{6.11}$$

This is the condition that may be solved to give back the original model.

7. Discussion

In this paper we have presented a path integral formulation of $\mathcal{N}=1$ supersymmetric PL duality. We have shown that it arises as a straightforward generalization of the treatment of the bosonic case. In this context we have also given an explicit representation of the generators of the Drinfel'd double bi-algebra corresponding to the group $G \times \mathrm{U}(1)^{\dim G}$ relevant for non-abelian dualization, and used it to elucidate the relation between the actions describing PL duality and the parent action in non-abelian $\mathcal{N}=1$ supersymmetric duality.

With applications to the $\mathcal{N}=2$ supersymmetric case in mind, we have also touched upon WZW models. In particular, in Appendix B we show how the treatment of the WZW model in [12] yields a formulation open also to PL duality.

The $\mathcal{N}=2$ supersymmetric case presents considerable difficulty, however. This is due to the severe constraints on the target manifold that such σ-models imply [21]. Our best approach so far to this problem is via an $\mathcal{N}=1$ superspace formulation where we may make use of the results in the present paper. Several questions such as the transformations of the complex and direct product structures under PL duality remain open however, and we have not found a complete characterization of the duality as yet. We hope to return to the question in a later publication.

The question of whether a PL type duality can be based on the extended currents that result when the Lie derivative of the background antisymmetric tensor field b

satisfies $\mathcal{L}b = d\omega$ may seem natural at a first encounter with PL duality. In Appendix A we discussed this question a little and found that under certain very special circumstances this can indeed be done. The answer is still essentially negative, though, high-lighting the special status of PL duality.

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A. A modified current and Poisson-Lie duality

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In this appendix we discuss some aspects of PL duality with modified currents.

Instead of isometries k_a that also preserve the b field $\mathcal{L}_{k_a} b_{ij} = 0$, we here turn to the case discussed in section 2: isometries that satisfy $\mathcal{L}_{k_a} b_{ij} = \partial_{[i}\omega_{j]a}$, (2.12). The question we address is whether it is possible to find a Maurer-Cartan equation as in (4.5), but for currents corresponding to (2.14) rather than (4.1).

We first note the following way of writing the Lie derivative of the b field:

$$\mathcal{L}_{k_a} b_{ij} = k_a^k H_{ijk} - \partial_{[i}(k_a^k b_{j]k})\,. \tag{A.1}$$

This shows that when k_a represent an invariance of the action and thus $k_a^k H_{ijk} = \partial_{[i}v_{j]a}$ (see below (2.13)), we may set

$$\omega_{ia} = v_{ia} - k_a^k b_{ik}\,, \tag{A.2}$$

This ensures integrability, i.e. $[\mathcal{L}_{k_a}, \mathcal{L}_{k_b}]b_{ij} = f_{ab}^c \mathcal{L}_{k_c} b_{ij}$.

We now turn to the general case and replace $k_a^i \to R_a^i$ as before. With this replacement (A.1) is still valid, but the action is no-longer taken to be invariant. We further assume that the b-field may be split according to

$$b_{ij} = \hat{b}_{ij} + b_{ij}^0\,, \tag{A.3}$$

where $\mathcal{L}_{R_a} b_{ij}^0 = \partial_{[i}\omega_{j]a}^0$, i.e. where the field strength H_{ijk}^0 of b_{ij}^0 is preserved by R_a^i.

The field equations (4.3) that result from the action (4.2) may then be rewritten as

$$\partial\hat{\bar{J}}_a + \bar{\partial}\hat{J}_a - \mathcal{L}_{R_a}\hat{f}_{ij}\partial x^i \bar{\partial} x^j = 0\,, \tag{A.4}$$

with the currents as in (2.14),

$$\hat{J}_a = \partial x^i (f_{ij} R_a^j + \omega_{ia}^0); \qquad \hat{\bar{J}}_a = (R_a^i f_{ij} - \omega_{ja}^0)\bar{\partial} x^j\,, \tag{A.5}$$

and

$$\hat{f}_{ij} = f_{ij} - b^0_{ij}\,. \tag{A.6}$$

In the general case we have found no modification of the PL condition satisfying the integrability conditions, which can turn the equation (A.4) into a Maurer-Cartan equation. Below we remark on a few special cases.

The first (trivial) case we consider is when $\omega^0_{ia} = 0$. The currents are then unmodified $\hat{J}_a = J_a$ and the PL condition is also unmodified

$$\mathcal{L}_{R_a}\hat{f}_{ij} = \mathcal{L}_{R_a} f_{ij} = -f_{ik}R^k_b\tilde{f}^{bc}_a R^l_c f_{lj}\,. \tag{A.7}$$

PL duality thus allows for a part of the b-field to be preserved by the Lie-derivative.

Second, we assume that v^0_{ia} in (A.2) may be gauged to zero using the b gauge transformations (cf comment below (2.12)). This means that $\omega^0_{ia} = R^k_a b^0_{ki}$ and this eliminates all dependence on b^0_{ij} in $\hat{J}$,$\hat{\bar{J}}$ and in $\hat{f}$. If we then *modify* the PL condition to read

$$\mathcal{L}_{R_a}\hat{f}_{ij} = -\hat{f}_{ik}R^k_b\tilde{f}^{bc}_a R^l_c\hat{f}_{lj}\,, \tag{A.8}$$

we find the Maurer-Cartan equations for the hatted currents:

$$\partial\hat{\bar{J}}_a + \bar{\partial}\hat{J}_a + \hat{J}_b\tilde{f}^{bc}_a\hat{\bar{J}}_c = 0\,. \tag{A.9}$$

(Integrability of (A.8) is ensured as before when $\mathcal{G}$ and $\tilde{\mathcal{G}}$ form a bi-Lie-algebra as in (4.7)). We see that we may indeed use the currents $\hat{J}$ and $\hat{\bar{J}}$ as a starting point for PL dualization in this case.

Finally we consider the case when $v^0_{ia} = R^k_a g_{ik}$ in (A.2). When R_a generate isometries and this relation holds for the whole b-field, the symmetry algebra is an infinite dimensional Kac-Moody algebra [13] and the currents reduce to one chiral current [10]. This latter fact results even if we do not assume isometries, since $\omega^0_{ia} = R^k_a f^0_{ki}$ which gives

$$\hat{J}_a = \partial x^i\left(2g_{ij} + (b - b^0)_{ij}\right)R^j_a, \quad \hat{\bar{J}}_a = -(b-b^0)_{ij}R^j_a\bar{\partial}x^i\,. \tag{A.10}$$

Clearly $J_a = 2g_{ij}\partial x^i R^j_a$ and $\bar{J}_a$ vanishes when $b_{ij} = b^0_{ij}$. In this case the PL condition applied to f_{ij} will lead to a PL dual model (expressed in $J, \bar{J}$, whereas $\hat{f}_{ij} = g_{ij}$ implies that the modified condition (A.8) vanishes (thus it only holds if we do have an isometry).

B. A comment on WZW-models and Poisson-Lie duality

The WZW model on G given in eq.(2.8) is not on the form used as starting point in the PL dualization scheme, because it is not on the form (2.4). It follows from (2.7) that

$$b_{ij} = \frac{1}{2}L^a_{[i}E_{ab}L^b_{j]}\,. \tag{B.1}$$

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Hence writing (2.8) in the form (2.4), given (2.10), requires the relation

$$\partial_{[i}(L^a_j E_{ab} L^b_{k]}) = \frac{1}{2} f_{abc} L^a_i L^b_j L^c_k\,, \tag{B.2}$$

to be satisfied. This leads to difficulties, at least for PL formulation of non-abelian duality (without spectators) where the E_{ab}'s must be constants.

In [12] when discussing non-abelian duality the following treatment of the WZW-model was given. Here we show how to adopt it for PL duality. Consider the WZW-model on $G \times G$. The action is written [12]

$$S_{G_k \times G_k}[g_1, g_2] = S_{G_k}[g_1] + S_{G_k}[g_2]\,, \tag{B.3}$$

where $S_{G_k}[g] = (k/2\pi) S[g]$; here $S[g]$ is defined in eq. (2.8). Next we define two new variables

$$g \equiv g_1\,; \qquad x \equiv g_2 g_1\,. \tag{B.4}$$

The element x is inert under the (gauge) transformation $g_1 \to u^{-1} g_1$, $g_2 \to g_2 u$. Using the Polyakov-Wiegmann formula

$$S_{G_k}[g_1 g_2] = S_{G_k}[g_1] + S_{G_k}[g_2] + \frac{k}{2\pi}\int d^2\xi\, \mathrm{Tr}(g_1^{-1}\partial g_1 \bar\partial g_2 g_2^{-1}) \tag{B.5}$$

and that the WZW term satisfies $\Gamma[g^{-1}] = -\Gamma[g]$ we rewrite the action $S_{G_k \times G_k}$ as

$$S_{G_k \times G_k}[g, x] = \frac{k}{2\pi}\int d^2\xi\, \mathrm{Tr}(g^{-1}\partial g g^{-1}\bar\partial g - x^{-1}\partial x g^{-1}\bar\partial g) + S_{G_k}[x]\,, \tag{B.6}$$

where $S_{G_k}[x]$ is an ordinary σ-model on a manifold with background $f_{\alpha\beta}(x)$. The action (B.6) can be compared with the original action found by integrating out the Lagrange multipliers λ_a of the action (3.3). Remembering the correspondence $x^{-1}\partial x = t_a L^a_\alpha \partial x^\alpha$ we see that $E_{ab} \equiv L^{\hat i}_a f_{\hat i \hat j} L^{\hat j}_b = \frac{k}{2\pi}\mathrm{Tr}(t_a t_b)$, $F^L_{\alpha b} = -\frac{k}{2\pi} L_\alpha{}^a\, \mathrm{Tr}(t_a t_b)$, $F^R_{a\beta} = 0$.

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